Vectors

A *vector* is an ordered list of numbers , as **Z = [2,4,6,8] which called row vector and**

A = $\begin{bmatrix} 3 \\ 5 \\ 76 \end{bmatrix}$ **which called column vector.**

Matrices

A *matrix* is a set of numbers ordered in rows and columns vectors, as in the example. Consider the 3 ×3 matrix $A = \begin{bmatrix} 2 & 5 & 6 \\ -6 & 7 & 9 \\ 12 & 55 & 13 \end{bmatrix}$

Note that the matrix *elements* in any row are separated by commas, and can also be separated by spaces.

Definition1:-Thesize of a matrix is given by(number of its rows x number of its columns).

Definition2:-Two matrices A , B are equal if and only if they are from the same size and the symmetric elements are equal , in other words A = B $\Leftrightarrow a_{ij} = b_{ij}\ \forall$ i=1,2,..,m and j=1,2,3,…..,n .

The general form of a matrix from size mxn is written as:-

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & \cdots\cdots & \cdots\cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & \cdots\cdots & a_{mn} \end{bmatrix}$$ or $A = [a_{ij}]$, i=1,2,3,….,m , j=1,2,3,….,n .

Kinds of matrices :-

1)Zero matrix (Null):- all its elements equal zero , denoted by O .
2)Square matrix is a matrix which its rows equal to its columns .
3)Lower triangular matrix is a square matrix which all $a_{ij} = 0$, $\forall$ $i < j$.
4)Upper triangular matrix is a square matrix which all $a_{ij} = 0$, $\forall$ $i > j$.
5)Diagonal matrix is a square matrix which all a_{ij} not lies on main diagonal equal to zero, And may be denoted as diag(a_{11},a_{22},a_{33},……….,a_{nn}) .
6)Scalar matrix is a diagonal matrix which diag(k,k,k,…..,k) such that k is a constant number .
7)Identity matrix is a diagonal matrix which diag(1,1,1,…..,1) and denoted by I .

Examples:-

1) $A = \begin{bmatrix} 6 & 5 & 42 \\ 0 & 2 & 8 \\ 6 & -4 & 9 \\ 8 & 11 & -10 \end{bmatrix}$ is a matrix from size 4 x 3 .

2) $B = \begin{bmatrix} 2 & 43 & 5 & 4 \\ 0 & 0 & -5 & 6 \\ 3 & 22 & 3.4 & \sqrt{6} \\ 2.45 & 89 & \pi & 8 \end{bmatrix}$ is a square matrix from size 4 .

3) $G = \begin{bmatrix} 2 & 0 & 0 \\ -6 & 7 & 0 \\ 12 & 55 & 13 \end{bmatrix}$ is l0wer triangular matrix from size 3 , and

$F = \begin{bmatrix} 2 & 5 & 6 \\ 0 & 7 & 9 \\ 0 & 0 & 13 \end{bmatrix}$ is upper triangular matrix from size 3 .

4) $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ is a diagonal matrix from size 4 .

5) $S = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ is a scalar matrix from size 4 .

6) $I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is Identity matrix from size 5 .

Operations on matrices :-

@1)*Addition of matrices* :- If two matrices **A** and **B** are from the same size, their (element-by-element) sum is obtained by typing **A + B** = C = $[c_{ij}]$, such that $c_{ij} = a_{ij} + b_{ij}, \forall$ i , j .
Likewise, the difference of **A** and **B** represents by **A – B**= D =$[a_{ij} - b_{ij}]$, $\forall$ i , j .

Example :- If $A = \begin{bmatrix} 6 & 5 \\ 7 & 0 \\ 8 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 5 \\ 8 & 3 \\ 14 & 7 \end{bmatrix}$, then $A + B = \begin{bmatrix} 8 & 10 \\ 15 & 3 \\ 22 & 8 \end{bmatrix}$.

$A - B = \begin{bmatrix} 4 & 0 \\ -1 & -3 \\ -6 & -6 \end{bmatrix}$, $B - A = \begin{bmatrix} -4 & 0 \\ 1 & 3 \\ 6 & 6 \end{bmatrix}$.

We can also add a scalar (a single number) to a matrix; **A + c** adds **c** to eachelement in **A**. Likewise, **A - c** subtracts the number **c** from eachelementof **A** , and c**A**multiply c by each element of **A** .

Example:- $A + 2 = \begin{bmatrix} 8 & 7 \\ 9 & 2 \\ 10 & 3 \end{bmatrix}$, $B - 4 = \begin{bmatrix} -2 & 1 \\ 4 & -1 \\ 10 & 3 \end{bmatrix}$, $5A = \begin{bmatrix} 30 & 25 \\ 35 & 0 \\ 40 & 5 \end{bmatrix}$.

Properties of matrices :-

For any matrices A , B , C , Zero matrix O from the same size ,and scalar numbers h , k :-
1) A + B = B + A .
2) A + (B + C) = (A + B) + C .
3) A + O = O + A = A .

4) A – A = O .
5) h(A + B) = hA + hB .
6) (h + k)A = hA + kA .
7) (hk)A = h(Ka) .
8) 1A = A , 0A = O .

@2) Multiplication of matrices :-

If **A** and **B** are multiplicatively compatible (that is, if **A** is $n\times m$ and **B** is $m\times p$), then their product **A*B** is $n\times p$. Recall that the element of **A*B** in the$i$throw and jth column is the sum of the products of the elements from the$i$throw of **A** times the elements from the $j$thcolumn of **B**, that is,

$(\mathbf{A} * \mathbf{B})_{ij}=\mathbf{A}_{ik}\mathbf{B}_{kj},\quad 1 \le i \le n,\quad 1 \le j \le p.$

Example :-

If $A = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 5 & 6 \\ 2 & -4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 & 0 & 3 \\ 2 & 11 & 8 & 1 \\ 6 & -2 & 4 & -1 \end{bmatrix}$, find AB , BA if possible .

***Solution**:-*

$a_{11} = [1 \ \ 3 \ \ -5]\begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$ = 1*5+3*2+(-5)*6 = -19 , $a_{12} = [1 \ \ 3 \ \ -5]\begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix}$ = 1*2+3*11+(-5)*(-2) = 45,

$a_{13} = [1 \ \ 3 \ \ -5]\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$ = 1*0+3*8+(-5)*4 = 4 , $a_{14} = [1 \ \ 3 \ \ -5]\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ = 1*3+3*1+(-5)*(-1) = 11,

$a_{21} = [0 \ \ 5 \ \ 6]\begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$ = 0*5+5*2+6*6 = 46 , $a_{22} = [0 \ 5 \ 6]\begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix}$ = 0*2+5*11+6*(-2) = 43 ,

$a_{23} = [0 \ 5 \ 6]\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$ = 0*0+5*8+6*4 =6 4 , $a_{24} = [0 \ 5 \ 6]\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ = 0*3+5*1+6*(-1) = -1,

$a_{31} = [2 \ \ -4 \ \ 7]\begin{bmatrix} 5 \\ 2 \\ 6 \end{bmatrix}$ = 2*5+(-4)*2+7*6 = 42 , $a_{32} = [2 \ \ -4 \ \ 7]\begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix}$ = 2*2+(-4)*11+7*(-2) = -54,

$a_{33} = [2 \ \ -4 \ \ 7]\begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$ = 2*0+(-4)*8+7*4 =- 4 , $a_{34} = [2 \ \ -4 \ \ 7]\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ = 2*3+(-4)*1+7*(-1) = -5,

$\therefore AB = \begin{bmatrix} -19 & 45 & 4 & 11 \\ 46 & 43 & 64 & -1 \\ 42 & -54 & -4 & -5 \end{bmatrix}$

BA is not possible because number of columns of B not equal to rows of A .

<u>Proposition</u>

1) If A is a square matrix from size n , then $AI_n = I_nA = A$.

2) If A(mxn) , B(nxp) , C(pxq) , then $A(BC) = (AB)C$.

Definition Let A is a matrix from size nxm , then the transpose of A is a matrix from size mxn denoted by A^T by changing rows with columns .

Example :-

If $A = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 5 & 6 \\ 2 & -4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 & 0 & 3 \\ 2 & 11 & 8 & 1 \\ 6 & -2 & 4 & -1 \end{bmatrix}$, find A^T , B^T .

$A^T = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & -4 \\ -5 & 6 & 7 \end{bmatrix}$, $B^T = \begin{bmatrix} 5 & 2 & 6 \\ 2 & 11 & -2 \\ 0 & 8 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

Proposition

1) If A , B two matrices from the same size then :- 11) $(A^T)^T = A$, 12) $(A+B)^T = A^T + B^T$
2) If A (mxn) , B(nxp) then $(AB)^T = B^T A^T$.

Example:-

Verify the proposition above for the matrices $A = \begin{bmatrix} 2 & 3 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 6 \\ 0 & 7 \end{bmatrix}$.

Definition:-

1)The square matrix A is called symmetric matrix if $A^T = A$, in other words $a_{ij} = a_{ji}$, $\forall\ i \neq j$.
2)The square matrix A is called skew-symmetric matrix if $A^T = -A$, in other words $a_{ij} = -a_{ji}$, $\forall\ i \neq j$ and the elements of main diagonal = 0 .

Examples:-

$A = \begin{bmatrix} 1 & 3 & -5 \\ 3 & 5 & 6 \\ -5 & 6 & 7 \end{bmatrix}$ is a symmetric matrix .

$F = \begin{bmatrix} 0 & -7 & 12 \\ 7 & 0 & -65 \\ -12 & 65 & 0 \end{bmatrix}$ is a skew-symmetric matrix .

Definition:- A square matrix A from size n is called (orthogonal matrix) if $AA^T = A^TA = I_n$.

Question :- Prove that $A = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$ is an orthogonal matrix ?

*Definition:-*Let A is a square matrix of size n , then the matrix B is called the invers matrix of A if and only if $AB = BA = I_n$ denoted by A^{-1}.

Notes

- Not for every square matrix an inverse .
- If A is an orthogonal matrix , then $A^T = A^{-1}$.

- IfA is a diagonal matrix such that D = $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$, then

$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix}$.

Question :- If A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}$ orthogonal matrix , find A^{-1} ?

Proposition

If A ,B are two square matrices of size n and they have inverse for them ,then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof :- $(AB).(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I_n$ …………………… (1)

$(B^{-1}A^{-1}).(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I_n$ …………………… (2)

$\therefore$ $(AB)^{-1} = B^{-1}A^{-1}$.

Question :- Prove that $(A^T)^{-1} = (A^{-1})^T$, if A is a square matrix and has an inverse ?

*Definition:-*For any square matrix A from size n there exist 0nly one number called the determinant of the matrix denoted by $|A|$.

Examples:-

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is a determinant from size 2 and its value = $a_{11} \cdot a_{22} - a_{21}a_{12}$.

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ is a determinant from size 3 and its value calculate as in the below :-

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$ = [main diagonals – secondary diagonals] =

$[a_{11}.a_{22}.a_{33}+a_{12}.a_{23}.a_{31}+a_{13}.a_{21}.a_{32}] - [a_{13}.a_{22}.a_{31}+a_{11}.a_{23}.a_{32}+a_{12}.a_{21}.a_{33}]$.

Definition :- The minor of $|A|$ is a determinant from $|A|$ after subtracting equal number from rows and columns of $|A|$.

The minor of a_{11} = $|M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ and the minor of a_{32} = $|M_{32}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

Definition :- The cofactor of the element $a_{ij} = (-1)^{i+j} . |M_{ij}|$ denoted by A_{IJ} .

Proposition

The value of any determinant equal to sum of multiplication elements of any rows(columns) by its cofactors .

Such that $|A| = a_{i1}A_{i1}+a_{i2}A_{i2}+a_{i3}A_{i3}+ \ldots\ldots\ldots\ldots +a_{in}A_{in}$, I = 1,2,3,………..,n OR

$= a_{1j}A_{1j}+a_{2j}A_{2j}+a_{3j}A_{3j}+ \ldots\ldots\ldots\ldots +a_{nj}A_{nj}$, j = 1,2,3,………..,n .

Example :- Find the value of $\begin{vmatrix} 1 & 4 & 9 & 2 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 7 \\ -3 & 0 & 9 & -2 \end{vmatrix}$

Proposition

If A is a square matrix and $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|} . [A_{ij}]^T$.

Example:- Find A^{-1} of $\begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 4 & 9 & 2 \\ 2 & 0 & 3 & 0 \\ 5 & 0 & 0 & 7 \\ -3 & 0 & 9 & -2 \end{vmatrix}$

Solving the system of linear equations by matrices

If we have the system of linear equations as below:-

$a_{11}x_1+a_{12}x_2+a_{13}x_3+ \ldots\ldots +a_{1n}x_n = b_1$

$a_{21}x_1+a_{22}x_2+a_{23}x_3+ \ldots\ldots +a_{2n}x_n = b_2$

$a_{31}x_1+a_{32}x_2+a_{33}x_3+ \ldots\ldots +a_{3n}x_n = b_3$

……………………………………………………..

……………………………………………………..

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$a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+ \ldots\ldots +a_{mn}x_n = b_m$

Then $A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & \cdots\cdots & \cdots\cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & \cdots\cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ b_m \end{bmatrix}$

∴ The solution to the system above is $X = A^{-1}.B$

Example :-

Solve the system of linear equations:-

2x+y=z

z-y+x=6

x+2y+z-3 =0

The solution

First :- we must arrange the equations as :-

2x+y-z=0

x-y+z=6
x+2y+z=3

A = $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B = $\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$

$|A| = \left|\begin{matrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{matrix}\right| \begin{matrix} 2 & 1 \\ 1 & -1 \\ 1 & 2 \end{matrix}$ = -9 ≠ 0

A_{11} = $\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$ = -3 , A_{12} = -1$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$ = 0 , A_{13} = $\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$ = 3

A_{21} = -1$\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$ = -3 , A_{22}= $\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$ = 3 , A_{23} = -1$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$ = -3

A_{31} = $\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$ = 0 , A_{32} = -1$\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$ = -3 , A_{33} = $\begin{vmatrix} 2 & 1 \\ 1 & - \end{vmatrix}$ = -3

$[A_{ij}]$ = $\begin{bmatrix} -3 & 0 & 3 \\ -3 & 3 & -3 \\ 0 & -3 & -3 \end{bmatrix}$, $[A_{ij}]^T$ = $\begin{bmatrix} -3 & -3 & 0 \\ 0 & 3 & -3 \\ 3 & -3 & -3 \end{bmatrix}$,

$\therefore\ A^{-1} = \frac{1}{-9}\begin{bmatrix} -3 & -3 & 0 \\ 0 & 3 & -3 \\ 3 & -3 & -3 \end{bmatrix}$

$\therefore\ X = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

Questions

1)If A = $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \\ 6 & 7 & -2 \end{bmatrix}$,B = $\begin{bmatrix} 9 & 0 & 12 \\ 5 & 6 & 1 \\ -1 & 0 & 0 \end{bmatrix}$, Find (1) 2A – 3B , (2) I_3 + 4B - 3B
(3) AB , BA , what do you notice ?

2) Write the matrix A from size 3x4 such that A = $\begin{cases} 7 & , \forall i < j \\ 3 & , \forall i = j \\ j - i & , \forall i > j \end{cases}$

3) Find the value of x , y , z , t if $\begin{bmatrix} 3x & 1 \\ z & 3+2t \end{bmatrix}$ - 2$\begin{bmatrix} x & -y \\ 3z & -2t \end{bmatrix}$ = 3$\begin{bmatrix} -1 & x-y \\ x+y & 2z \end{bmatrix}$

4) If A = $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 3 & -2 \\ 2 & -3 & 5 \end{bmatrix}$, B = $\begin{bmatrix} -11 \\ -16 \\ 21 \end{bmatrix}$, find the matrix X which satisfy AX = B .

5) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & 1/2 \end{bmatrix}$ is orthogonal matrix , find A^{-1} .

6) Write a symmetric matrix from size 4 .

7) Write a skew-symmetric matrix from size 5 .

8) Is there exist an inverse matrix for $A = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$?

9) S0lve the following system of linear equations:-

(1)
$$2x_1 - 4x_2 - x_3 = 2$$
$$3x_2 - 2x_3 + x_1 = 0$$
$$-6 + 3x_1 = 2x_2 + 3x_3$$

(2)
$$10x_3 + 6x_2 = 9 - 3x_1$$
$$x_1 + x_2 = 4 - x_3$$
$$3x_2 + 2x_1 + 4x_3 = 0$$

10) Find the value of $\begin{vmatrix} 4 & 5 & -6 & -1 \\ 2 & 8 & 5 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{vmatrix}$

11) If $A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 0 \\ 11 & 8 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 7 \\ 9 & 2 \\ 10 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 90 \\ 2 & 4 \\ -4 & 12 \end{bmatrix}$, Is A .(B + C) = A.B + A.C ?

12) Full the following statements with suitable words :-

a) Tow matrices are equal if and only if ………………….. .

b) Lower triangular matrix is ……………………………….. .

c) Upper triangular matrix is ……………………………….. .

d) Diagonal matrix is ………………………………………. .

e) We can multiply the matrix A by the matrix B if ……. .

f) A is a symmetric matrix if and only if ……………………. .

g) IF the matrix $A = -A^T$ then A is called………………….. .

h) If $AB = BA = I_n$ then B is called ………………………… .

t) The minor to a_{ij} of the determinant A is ………………….

k) The cofactor of the element a_{ij} denoted by …….. and equal to …………. .

13) For the matrices $A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 0 \\ 11 & 8 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & -4 \\ -5 & 6 & 7 \end{bmatrix}$

Verify the following :-

1- A + B = B + A
2- A + (B + C) = (A + B) + C
3- A + O = O + A = A
4- C – C = O
5- $(B^T)^T = B$
6- $AI_3 = I_3A = A$
7- A(BC) = (AB)C
8- $(B + C)^T = B^T + C^T$

Functions and differentiation

DEFINITION :- The function y = f(x) is said to be a differentiable function of x if $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ exist and finite .

DERIVATIVE:- We can find the derivative to the function y = f(x) by using the definition and denoted by :-

$y' = f'(x) = \frac{dy}{dx}$ which called the first derivative and ,

$y'' = f''(x) = \frac{d^2y}{dx^2}$ which called the second derivative

$y''' = f'''(x) = \frac{d^3y}{dx^3}$ which called the third derivative

And so on .

EXAMPLES :- Find the derivatives for the following by definition:-

1) F(x) = x-7
2) G(x) = x^2 +2x-4
3) Y = $\sqrt{5-x}$
4) Y = $\frac{4}{x}$

1)F'(x) = $\lim_{\Delta x \to 0} \frac{(2x+2\Delta x-7)-(2x-7)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x+2\Delta x-7-2x+7}{\Delta x}$ = 2 .

3)y'= $\lim_{\Delta x \to 0} \frac{\sqrt{5-x-\Delta x}-\sqrt{5-x}}{\Delta x}$ =

$\lim_{\Delta x \to 0} \frac{\sqrt{5-x-\Delta x}-\sqrt{5-x}}{\Delta x} * \frac{\sqrt{5-x-\Delta x}+\sqrt{5-x}}{\sqrt{5-x-\Delta x}+\sqrt{5-x}} = \frac{-1}{2\sqrt{5-x}}$

STANDARD DIFFERENTIAL COEFFICIENTS

Functions	Derivatives
1- Y = c	$\frac{dy}{dx} = 0$
2- Y = x^n	$\frac{dy}{dx} = nx^{n-1}$
3- Y = cx^n	$\frac{dy}{dx} = ncx^{n-1}$
4- Y = f(x) $\mp$ g(x)	$\frac{dy}{dx} = f'(x) \mp g'(x)$
5- Y = f(x).g(x)	$\frac{dy}{dx} = f(x).\, g'(x) + g(x).f'(x)$
6- Y = $\frac{f(x)}{g(x)}$, g(x) ≠ 0	$\frac{dy}{dx} = \frac{g(x).f'(x) - f(x).\, g'(x)}{[g(x)]^2}$
7- Y = $[f(x)]^n$	$\frac{dy}{dx} = n[f(x)]^{n-1}.f'(x)$

EXAMPLES:- Find the derivatives for the following :-

1) Y = $x^5 + 3x^3 - \frac{3}{5}x^2 + \frac{2}{x} + 54 \Rightarrow \frac{dy}{dx} = 5x^4 + 9x^2 - \frac{6}{5}x - 2x^{-2}$

2) Y = $(4x^3 - \frac{7}{8}x)(4+x/7) \Rightarrow \frac{dy}{dx} = (4x^3 - \frac{7}{8}x)*(1/7) + (4+x/7)*(12x^2 - \frac{7}{8})$

3) F(x) = $\frac{\sqrt[5]{3x+6}}{(x^3 - 2x^2)} \Rightarrow \frac{\frac{1}{5}}{(x^3 - 2x^2)}$

4) U = $\sqrt[5]{\frac{3x^3 + 5x^2}{89x - 23}}$

THE CHAIN RULE:-

If y = f(u) and u = g(x) then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$

EXAMPLES:- Find $\frac{dy}{dx}$ for the following :-

1) Y = $u^6 + \sqrt{3u}$, u = $(2x - x^3) \Rightarrow \frac{dy}{dx} = \left(6u^5 + \frac{3}{2\sqrt{3u}}\right).(2 - 3x^2)$

$$\Rightarrow \frac{dy}{dx} = \left(6(2x - x^3)^5 + \frac{3}{2\sqrt{3(2x - x^3)}}\right).(2 - 3x^2)$$

2) Y = $n^3 + 6$, n = x(x+2) , then $\frac{dy}{dx} = \frac{dy}{dn}.\frac{dn}{dx} = (3n^2).(2x+2)$

$=3(x^2+2x)^2(2x+2)=6(x^4+4x^3+4x^2)(x+1)=6(x^5+5x^4+8x^3+4x^2)$

THE DERIVATIVE OF TRIGONOMETRIC FUNCTIONS:-

function	derivative
1- Y= sinu	$\frac{dy}{dx}$ = cosu. $\frac{du}{dx}$
2- Y= cosu	$\frac{dy}{dx}$ = -sinu. $\frac{du}{dx}$
3- Y= tanu	$\frac{dy}{dx}$ = $\sec^2$u. $\frac{du}{dx}$
4- Y= cotu	$\frac{dy}{dx}$ = $-\csc^2$u. $\frac{du}{dx}$
5- Y= secu	$\frac{dy}{dx}$ = secu.tanu. $\frac{du}{dx}$
6- Y= cscu	$\frac{dy}{dx}$ = -cscu.cotu. $\frac{du}{dx}$

EXAMPLES:- Find $\frac{dy}{dx}$ for the following :-

1) $Y= x^3\sin4x$
2) $Y= \tan^2x + \sin\sqrt{x}$
3) $Y= \cos3u$, $u= \sec(4x- 7)$
4) $Y= \sqrt[3]{\csc 7x.\cot(x/6)}$

THE SOLUTION:-

1) $\frac{dy}{dx} = x^3.4\cos4x + \sin4x.3x^2 = 4x^3\cos4x + 3x^2.\sin4x$

2) $\frac{dy}{dx} = 2\tan x + \cos\sqrt{x} . \frac{1}{2\sqrt{x}} = 2\tan x + \frac{1}{2\sqrt{x}}\cos\sqrt{x}$

3) $\frac{dy}{dx} = [-3\sin3u].[\sec(4x-7).\tan(4x-7).4] = -12\sin[3\sec(4x-7)]. \sec(4x-7).\tan(4x-7)$

4) $\frac{dy}{dx} = \frac{1}{3}.$
$\sqrt[3]{[\csc 7x.\cot(x/6)]^{-2}}[\csc 7x\{-\csc(x/6)\cot(x/6)(1/6)\} + \cot(x/6)\{-7\csc 7x.\cot 7x\}]$

Now here is a short set for you to do .Find $\frac{dy}{dx}$ when

1. $Y= x^2\tan x$
2. $Y= (3x+1)/\sec\sqrt{3x}$
3. $Y= x^5\cos x^2\csc6/x$

IMPLICIT FUNCTIONS:-

If $y= x^2-4x+3$, y is completely defined in terms of x and y is called an explicit function of x .

When the relationship between x and y is more involved, it may not be possible to separate y completely on the left-side , e.g. xy+siny=3. In such a case as this , y is called implicit function of x, because a relationship of the form y=f(x) is implicit in the given equation.

It may still be necessary to determine the differential coefficients of y with respect to x and in fact this is not all difficult. All we have to remember is that y is a function of x, even if it is difficult to see what is it, this is really extension of our "function to function" routine.

$x^2+y^2=25$ is an example of an implicit function, if we differentiate it with respect to x, we get

$2x+2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x \qquad \therefore \frac{dy}{dx} = -\frac{x}{y}$

EXAMPLE 1.

If $x^2 + y^2 - 2x - 6y + 5 = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 3 , y = 2.

$2x+2y\frac{dy}{dx} - 2 - 6\frac{dy}{dx} = 0$

$\therefore\ (2y - 6)\dfrac{dy}{dx} = 2 - 2x$

$\therefore\ \dfrac{dy}{dx} = \dfrac{2-2x}{2y-6} = \dfrac{1-x}{y-3}$

$\therefore$ at (3,2) $\dfrac{dy}{dx} = \dfrac{1-3}{2-3} = \dfrac{-2}{-1} = 2$

Then $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left\{\dfrac{1-x}{y-3}\right\} = \dfrac{(y-3)(-1)-(1-x)\dfrac{dy}{dx}}{(y-3)^2}$

$= \dfrac{(3-y)-(1-x)\dfrac{dy}{dx}}{(y-3)^2}$

$\therefore$ at (3,2) $\dfrac{d^2y}{dx^2} = \dfrac{(3-2)-(1-3)2}{2-3^2} = \dfrac{1-(-4)}{1} = 5$

$\therefore$ at (3,2) $\dfrac{dy}{dx} = 2$, $\dfrac{d^2y}{dx^2} = 5$

EXAMPLE 2.

If $x^2 + 2xy + 3y^2 = 7$, find $\dfrac{dy}{dx}$ at x = 10 , y = 5.

$2x + 2x\dfrac{dy}{dx} + 2y + 6y\dfrac{dy}{dx} = 0$

$(2x + 6y)\dfrac{dy}{dx} = -(2x + 2y)$

$\therefore\ \dfrac{dy}{dx} = -\dfrac{2x+2y}{2x+6y} = -\dfrac{x+y}{x+3y}$

$\therefore$ at (10,5) $\dfrac{dy}{dx} = -\dfrac{10+5}{10+15} = -\dfrac{15}{25} = -\dfrac{3}{5}$

FURTHER DIFFERENTIATION:-

(1) NATURAL LOGARITHM :- $\dfrac{d}{dx}\{\ln u\} = \dfrac{1}{u}.\dfrac{du}{dx}$

(2) ORDINARY LOGARITHM :- $\dfrac{d}{dx}\left\{\log_a u\right\} = \dfrac{1}{u.\ln a}.\dfrac{du}{dx}$

(3) EXPONENTIAL FUNCTION :-

1- $\dfrac{d}{dx}\{e^u\} = e^u.\dfrac{du}{dx}$, e = 2.718…..

2- $\dfrac{d}{dx}\{a^u\} = a^u.\ln a.\dfrac{du}{dx}$, a > 0

(4) If $y = u^v$ $\Rightarrow \dfrac{dy}{dx} = \left[\dfrac{u}{v}.\dfrac{du}{dx} + \ln u.\dfrac{dv}{dx}\right]$

Examples:- find $\frac{dy}{dx}$ for the following:-

1) $Y = e^{3-x}$
2) $Y = \ln(3+4\cos x)$
3) $Y = 6^{\sin 4x}$
4) $Y = \log_{10}(2x-1)$
5) $Y = \dfrac{x^2 \sin x}{\cos 2x}$

The solution :-

1) $\frac{dy}{dx} = e^{3-x}(-1) = -e^{3-x}$
2) $\frac{dy}{dx} = \dfrac{1}{(3+4\cos x)}.(-4\sin x) = \dfrac{-4\sin x}{3+4\cos x}$
3) $\frac{dy}{dx} = 6^{\sin 4x}.\ln 6.4\cos 4x$
4) $\frac{dy}{dx} = \dfrac{1}{(2x-1).\ln 10}.(2) = \dfrac{2}{(2x-1)}$
5) $\ln y = \ln(x^2) + \ln(\sin x) - \ln(\cos 2x)$, now diff.both sides with respect to x

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{x^2}.2x + \frac{1}{\sin x}.\cos x - \frac{1}{\cos 2x}.(-2\sin 2x)$$

$$= \frac{2}{x} + \cot x + \tan 2x$$

$$\therefore \frac{dy}{dx} = \frac{x^2 \sin x}{\cos 2x}\left\{\frac{2}{x} + \cot x + \tan 2x\right\}$$

Now you can differentiate this one as above   If $y = x^4 e^{3x} \tan x$, then $\frac{dy}{dx}$ = …….

FURTHER PROBLEMS:-

1)If $x^2 + y^2 - 2x + 2y = 23$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point where $x = -2$, $y = 3$.

2)If $x= 3(1-\cos\theta)$ and $y= 3(\theta-\sin\theta)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in their simplest forms.

3)Find $\frac{dy}{dx}$ for the following equations:-

a- $x^3 + y^3 + 4xy^2 = 5$

b- $y = \dfrac{x\sin x}{1+\cos x}$

c- $y = \ln\left\{\dfrac{1-x^2}{1+x^2}\right\}$

d- $y = e^{\sin^2 5x}$

4) $y = a\sin^3\theta$, $x = a\cos^3\theta$

5) If $(x-y)^3 = A(x+y)$, prove that $(2x+y)\frac{dy}{dx} = x+2y$.

6) If $x^2 + 2xy + 3y^2 = 1$, prove that $(x+3y)^3 \frac{d^2y}{dx^2} + 2 = 0$.

7) If $x = \ln(\tan\frac{\theta}{2})$ and $y = \tan\theta - \theta$ prove that $\frac{d^2y}{dx^2} = \tan^2\theta.\sin\theta(\cos\theta + 2\sec\theta)$.

8) If $x = 3e^{2x}\cos(2x-3)$, verify that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 8y = 0$.

9) If $y = \left\{x + \sqrt{(1+x^2)}\right\}^{3/2}$, show that $4(1+x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 9y = 0$.

10) Show that $y = e^{-2mx}\sin 4mx$ is a solution of the equation $\frac{d^2y}{dx^2} + 4m\frac{dy}{dx} + 20m^2y = 0$.

11) If $y = \sec x$, prove that $y\frac{d^2y}{dx^2} = (\frac{dy}{dx})^2 + y^4$.

12) Prove that $x = Ae^{-kt}\sin pt$, satisfies the equation $\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + (p^2+k^2)x = 0$.

13) Differentiate :-

i) $y = \ln\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}$

ii) $y = x^2\cos^2 x$

iii) $y = \frac{4^{2x}\ln 7x}{(x-1)^{(3)}}$

14) If $x = \frac{2-3t}{1+t}$, $y = \frac{3+2t}{1+t}$, find $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$.

15) find $\frac{dy}{dx}$ for the function $y = x^5\sin x.\csc 4x$.

16) Prove that :- 1) $\frac{d}{dx}\left(\frac{\sin x}{a+b\cos x}\right) = \frac{a\cos x + b}{(a+b\cos x)^2}$

2) $\frac{d}{dx}\left(\sqrt{1-c^2\cdot\sin^2 x}\right) = \frac{-c^2.\sin 2x}{2\sqrt{1-c^2\cdot\sin^2 x}}$

3) $\frac{d}{dx}\left(\frac{1-\cos 2x}{1+\cos 2x}\right) = 2\tan x.\sec^2 x$

4) $\frac{d^2}{dx^2}(\cot x) = 2\cot x.\csc^2 x$

PARTIAL DIFFERENTIATION:-

In the statement , V= $\pi r^2 h$, V is expressed as a function of two variables, r and h .It therefore has two partial differential coefficients, one with respect to x and one with respect to h.

Another Example

Let us consider the area of the curved surface of the cylinder.

A = 2πrh

A is a function of r and h, so we can find $\frac{\partial A}{\partial r}$ and $\frac{\partial A}{\partial h}$

To find $\frac{\partial A}{\partial r}$ we differentiate the expression for A with respect to r, keeping all other symbols constant.

To find $\frac{\partial A}{\partial h}$ we differentiate the expression for A with respect to h, keeping all other symbols constant.

So, if A = 2πrh, then $\frac{\partial A}{\partial r}$ =……….. and $\frac{\partial A}{\partial h}$ =………….. .

Example :- consider Z = x^2y^3 .

1) To find $\frac{\partial Z}{\partial x}$,differentiate w.r.t. x, regarding y as a constant.

$$\therefore \frac{\partial Z}{\partial x} = 2xy^3$$

2) To find $\frac{\partial Z}{\partial y}$,differentiate w.r.t. y, regarding x as a constant.

$$\therefore \frac{\partial Z}{\partial y} = 3x^2y^2$$

Partial differentiation is easy! For we regard every independent variable, except the one with respect to which we are differentiating, as being for the time being …

Here are two examples:-

Example 1. U= x^2+xy+y^2

$$\therefore \frac{\partial U}{\partial x} = 2x + y \qquad \text{(y is a constant)}$$

$$\therefore \frac{\partial U}{\partial x} = x + 2y \qquad \text{(x is a constant)}$$

Example 2.

If Z = (4x-2y)(3x+5y) , find $\frac{\partial Z}{\partial x}$ and $\frac{\partial Z}{\partial y}$.

The solution:- Z= $12x^2 + 14xy - 10y^2$

$$\therefore \frac{\partial Z}{\partial x} = 24x + 14y \qquad \text{and} \qquad \frac{\partial Z}{\partial y} = 14x - 20y \ .$$

Example 2. If $Z = \frac{2x-y}{x+y}$ find $\frac{\partial Z}{\partial x}$ and $\frac{\partial Z}{\partial y}$.

The solution:-

$$\therefore \frac{\partial Z}{\partial x} = \frac{(x+y)(2)-(2x-y)(1)}{(x+y)^2} = \frac{3y}{(x+y)^2} \quad \text{and}$$

$$\frac{\partial Z}{\partial y} = \frac{(x+y)(-1)-(2x-y)(1)}{(x+y)^2} = \frac{-3x}{(x+y)^2}$$

Now you do these exercises to find $\frac{\partial Z}{\partial x}$, $\frac{\partial Z}{\partial t}$ and $\frac{\partial Z}{\partial y}$:-

1) $Z = \frac{5x+y}{x-2y} + \sec 8t$
2) $Z = \sin(3x+2y) - x^3yt^6$
3) $Z = \frac{\sin(2x-y)}{xy} + \tan^3 t$

PARTIAL DIFFERENTIATION FROM HIGHER ORDERS:-

Consider $Z = 3x^2 + 4xy - 5y^2$, then $\frac{\partial Z}{\partial x} = 6x + 4y$ and $\frac{\partial Z}{\partial y} = 4x - 10y$

The expression $\frac{\partial Z}{\partial x} = 6x + 4y$ is it self a function of x , y.We could therefore find its partial differential coefficients with respect to x or to y .

(1) If we differentiate it partially w.r.t. x, we get:- $\frac{\partial}{\partial x}\left\{\frac{\partial z}{\partial x}\right\}$ and this is written as $\frac{\partial^2 Z}{\partial x^2}$. $\therefore \frac{\partial^2 Z}{\partial x^2} = \frac{\partial}{\partial x}(6x+4y) = 6$ .This is called the second partial differential coefficient of Z with respect to x.

(2) If we differentiate it partially w.r.t. y, we get:- $\frac{\partial}{\partial y}\left\{\frac{\partial z}{\partial x}\right\}$ and this is written as $\frac{\partial^2 Z}{\partial y \partial x}$. $\therefore \frac{\partial^2 Z}{\partial y \partial x} = \frac{\partial}{\partial y}\left\{\frac{\partial Z}{\partial x}\right\} = \frac{\partial}{\partial y}\{6x+4y\} = 4$.

Of course , we could carry out similar steps with the expression for $\frac{\partial Z}{\partial y}$ on the right .

This would give us : $\frac{\partial^2 Z}{\partial y^2} = -10$, $\frac{\partial^2 Z}{\partial x \partial y} = 4$.

Note that $\frac{\partial^2 Z}{\partial y \partial x}$ means $\frac{\partial}{\partial y}\left\{\frac{\partial z}{\partial x}\right\}$, so $\frac{\partial^2 Z}{\partial x \partial y}$ means ....

HEAR IS ONE FOR YOU TO DO:-

If $z = 5x^3 + 3x^2y + 4y^5$, find $\frac{\partial Z}{\partial x}$, $\frac{\partial Z}{\partial y}$, $\frac{\partial^2 Z}{\partial y \partial x}$, $\frac{\partial^2 Z}{\partial x \partial y}$, $\frac{\partial^2 Z}{\partial x^2}$, $\frac{\partial^2 Z}{\partial y^2}$.

FURTHER PROBLEMS:-

1) If $V = \ln(x^2+y^2)$, prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.

2) If $V = f(x^2+y^2)$, prove that $x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x} = 0$.

3) If $V = x^2 + y^2 + z^2$, express in its simplest form $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$

4) If $Z = \sin(3x+2y)$, verify that $3\frac{\partial^2 Z}{\partial y^2} - 2\frac{\partial^2 Z}{\partial x^2} = 6z$.

5) If $u = \frac{x+y+z}{\left(x^2+y^2+z^2\right)^{1/2}}$ m show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ z = e = 0 .

6) If $Z = e^x(x\cos y - y\sin y)$, show that $\frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} = 0$.

7) If $f = \frac{1}{\sqrt{1-2xy+y^2}}$, show that $y\frac{\partial f}{\partial y} = (x-y)\frac{\partial f}{\partial x}$.

8) Find $\frac{dy}{dx}$ given that $x^2y + \sin xy = 0$.

9) Find $\frac{dy}{dx}$ given that $x\sin xy = 1$.

10) Prove that , if $z = 2xy + x.f\left(\frac{y}{x}\right)$ then $x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = z + 2xy$.

11) If $W = 2x^2 + 4xy + 7z$, find W_x , W_Y , W_Z , W_{XX} , W_{XY} , W_{YX} , W_{YY} .

12) If $W = \frac{x^2y^2}{x+y}$ prove that $xW_X + YW_Y = 3W$.

13) If $f(x,y) = \log_5 \sqrt{x^2+y^2}$, prove that $f_{xx} + f_{yy} = 0$.

Notice that

1) $W_x = \frac{\partial w}{\partial x}$

2) $W_Y = \frac{\partial w}{\partial y}$

3) $W_{XX} = \frac{\partial^2 w}{\partial x^2}$

4) $W_{XY} = \frac{\partial^2 w}{\partial x \partial y}$, $W_{YX} = \frac{\partial^2 w}{\partial y \partial x}$

5) $W_{YY} = \frac{\partial^2 Z}{\partial y^2}$

INTEGRATION

Every differential coefficient, when written in reverse, gives us an integral,

e.g. $\frac{d}{dx}(\sin x) = \cos x \qquad \therefore \int \cos x dx = \sin x + c$

THERE ARE TWO KINDS OF INTEGRATION

- Indefinite integration like $\int (f(x))' dx = f(x) + c$
- Definite integration like $\int_a^b y dx = [F(x)]_a^b = F(b) - F(a)$

Here is a list of basic differential coefficients and the basic integrals that go with them:

1. $\frac{d}{dx}[f(x) \mp g(x)] = \frac{d}{dx}[f(x)] \mp \frac{d}{dx}[g(x)]$	$\int [f(x) \mp g(x)]dx = \int f(x)dx \mp g(x)dx$
2. $\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$
3. $\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$	$\int u^{-1} du = \ln u + c$
4. $\frac{d}{dx}(e^u) = e^u. \frac{du}{dx}$	$\int e^u du = e^u + c$
5. $\frac{d}{dx}(a^u) = a^u. \ln a. \frac{du}{dx}$	$\int a^u du = \frac{a^u}{\ln a} + c$
6. $\frac{d}{dx}(\sin u) = \cos u. \frac{du}{dx}$	$\int \sin u du = -\cos u + c$
7. $\frac{d}{dx}(\cos u) = -\sin u. \frac{du}{dx}$	$\int \cos u du = \sin u + c$
8. $\frac{d}{dx}(\tan u) = \sec^2 u. \frac{du}{dx}$	$\int \sec^2 u du = \tan u + c$
9. $\frac{d}{dx}(\cot u) = -\csc^2 u. \frac{du}{dx}$	$\int \csc^2 u du = -\cot u + c$
10. $\frac{d}{dx}(\sec u) = \sec u. \tan u. \frac{du}{dx}$	$\int \sec . \tan u du = \sec u + c$
11. $\frac{d}{dx}(\csc u) = -\csc u. \cot u. \frac{du}{dx}$	$\int \csc u . \cot u du = -\csc u + c$
12. $\frac{d}{dx}[f(x)]^n = [f(x)]^{n-1}. \frac{df}{dx}$	$\int [f(x)]^n . f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$
13. $\frac{d}{dx}[f(x) * g(x)] = f(x) \frac{dg}{dx} + g(x) \frac{df}{dx}$	$\int [f(x)]^{-1}). f'(x)dx = \ln[f(x)] + c$
14. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{[g(x)]^2}$	$\int \frac{f'(x)}{f(x)} dx =$ $\int [f(x)]^{-1}). f'(x)dx = \ln[f(x)] + c$

EXAMPLES:-

1) $\int e^{5x}dx = \frac{e^{5x}}{5} + c$

2) $\int (x^7 + 4x^3 - 2)dx = \frac{x^8}{8} + x^4 - 2x + c$

3) $\int \sqrt{x}dx = 2\frac{x^{3/2}}{3} + c$

4) $\int \frac{5}{x}dx = 5\ln x + c$

5) $\int 5^x dx = \frac{5^x}{\ln 5} + c$

6) $\int (5x-4)^6 dx$, in this example we suppose that u = 5x-4 → du = 5dx ,

$\therefore$ dx = $\frac{1}{5}du$ and the integral becomes $\int u^6 \frac{1}{5}du = \frac{1}{5}\int u^6 du = \frac{1}{5}\frac{u^7}{7} + c$

$= \frac{(5x-4)^7}{35} + c$

This will always happen when we integrate functions of a linear function of x.

e.g. $\int e^x dx = e^x + c$ $\quad \therefore \int e^{3x+4}dx = \frac{e^{3x+4}}{3} + c$

Similarly, since $\int \cos x dx = \sin x + c$ $\quad \therefore \int \cos(2x-7)dx = \frac{\sin(2x-7)}{2} + c$

$\int \sec^2 x dx = \tan x + c$ $\quad \therefore \int \sec^2(9+4x)dx = \frac{\tan(9+4x)}{4} + c$

$\int 2^x dx = \frac{2^x}{\ln 2} + c$ $\quad \therefore \int 9^{3x+4}dx = \frac{9^{3x+4}}{3\ln 9} + c$

Now you can do these quite happily , and do not forget the constants of integration!

1. $\int (7-2x)^5 dx$ 2. $\int \sin(6x-1)dx$ 3. $\int \csc(4x-10)\cot(4x-10)dx$

4. $\int \frac{1}{3x-7}dx$ 5. $\int \cot^2(1-7x)dx$ 6. $\int (10)^{3x+5}dx$

INTEGRALS OF THE FORM $\int \frac{f'(x)}{f(x)}dx$ and $\int f(x)\, f'(x)dx.$

1) Consider the integral $\int \frac{2x+3}{x^2+3x-5}dx$, we notice that if we differentiate the denominator , we obtain the expression in the numerator. So, let z= x^2+3x-5 ,

$\frac{dz}{dx} = 2x + 3$, $\therefore$ dz = (2x+3)dx

$\therefore$ The given integral can then be written in the terms of z .

$\therefore \int \frac{2x+3}{x^2+3x-5}dx = \int \frac{dz}{z} = \ln z + c = \ln(x^2+3x-5) + c$

2) Consider the integral $\int \frac{2x+3}{\sqrt{(x^2+3x-5)}}dx$,

let z= x^2+3x-5 , $\frac{dz}{dx} = 2x+3$, ∴ dz = (2x+3)dx

∴ The given integral can then be written in the terms of z .

$$\int \frac{2x+3}{\sqrt{(x^2+3x-5)}}dx = \int \frac{dz}{\sqrt{z}} = \int z^{-1/2}dz = \frac{z^{1/2}}{1/2} + c = 2\sqrt{x^2+3x-5} + c$$

3) Consider the integral $\int \frac{2x^2}{\sqrt[3]{\sqrt{(x^3+12)}}}dx$,

First we simplify the integral which becomes $2\int (x^3+12)^{-1/6} x^2 dx$

Let z = x^3+12 , dz = $3x^2dx$ ⟹ x^2dx = dz/3

∴ The given integral can then be written in the terms of z .

$$2\int (x^3+12)^{-1/6} x^2 dx = 2\int z^{-1/6}\frac{dz}{3} = \frac{2}{3}\int z^{-1/6}dz = \frac{2}{3}*\frac{6}{5}z^{5/6} + c = \frac{4}{5}(x^3+12)^{5/6} + c$$

4) Consider the integral $\int \tan x dx$

∵ tanx = $\frac{\sin x}{\cos x}$

∴ $\int \tan x dx = -\int \frac{-\sin x}{\cos x}dx = -\ln(\cos x) + c$

Now you must complete the following:-

1. $\int \frac{\sec^2 x}{\tan x}dx = ...$

2. $\int \frac{2x+4}{x^2+4x-1}dx = ...$

3. $\int \frac{x-3}{\sqrt[3]{(x^2-6x+2)^4}}dx = ...$

4. $\int \tan 2x.\sec^2 2x.dx = ...$

5. $\int \sin x.\cos x dx = ...$

6. $\int \frac{\ln x}{x}dx = ...$

7. $\int \frac{\cos x}{1+\sin x}dx = ...$

8. $\int \sin^2 x dx = ...$

9. $\int \tan^2 x dx = ...$

INTEGRATION OF PRODUCT – INTEGRATION BY PARTS

We often need to integrate a product where either function is not the differential coefficient of the other. For example, in the case of $\int x^2 . \ln x dx,$

Lnx is not the differential coefficient of x^2
X^2 is not the differential coefficient of lnx

Let us establish the rule for such cases.

If u and v are functions of x, then we know that $\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$, by integrate both sides with respect to x , we get uv = $\int u\frac{dv}{dx}dx + \int v\frac{dv}{dx}dx$, and rearranging the terms, we have $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

For convenience, this can be memorized as $\int udv = uv - \int vdu$.

This method is called integration by parts.

EXAMPLE 1.

$$\int x^2 . \ln x dx$$

Let u = lnx then du = $\frac{1}{x}dx$ and dv = x^2dx then v= $\frac{x^3}{3}$

$$\therefore \int x^2 . \ln x dx = \ln x (\frac{x^3}{3}) - \frac{1}{3}\int x^3 . \frac{1}{x} dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\left\{\ln x - \frac{1}{3}\right\} + c$$

EXAMPLE 2.

$$\int x^2 \cdot e^{3x} dx$$

Let u = x^2 then du = 2xdx and dv = e^{3x}dx then v = $\frac{1}{3}e^{3x}$, then

$$\int x^2 \cdot e^{3x} dx =$$

$$\frac{x^2 \cdot e^{3x}}{3} - \frac{2}{3}\int e^{3x} . xdx = \frac{x^2 \cdot e^{3x}}{3} - \frac{2}{3}\left\{ x\left(\frac{e^{3x}}{3}\right) - \frac{1}{3}\int e^{3x} dx \right\}$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2x e^{3x}}{9} + \frac{2}{9} . \frac{e^{3x}}{3} + c = \frac{e^{3x}}{3}\left\{ x^2 - \frac{2x}{3} + \frac{2}{9} \right\} + c$$

Now determine the following integrals:-

1. $\int x . \ln x dx$
2. $\int x^3 . \sin x dx$
3. $\int e^{4x} . \cos x dx$
4. $\int x^3 . \ln(x+4) dx$
5. $\int (x-1)^2 . \ln x dx$
6. $\int \frac{1-\sin\theta}{\cos^2\theta} d\theta$ ________________ 7. $\int \frac{\sin 2x}{1+\cos^2 x} dx$ solve

INTEGRATION BY PARTIAL FRACTIONS

Suppose we have $\int \frac{x+1}{x^2-3x+2}dx$. Clearly this is not one of our standard types, and the numerator is not the differential coefficient of the denominator .So that we integrate it by (partial fractions method).

THE RULES OF PARTIAL FRACTIONS ARE AS FOLLOWS:-

- The numerator of the given function must be of lower degree than that of the denominator. If it is not , then first of all divide out by long division.
- Factorize the denominator into its prime factors. This is important since the factors obtained determine the shape of the partial fractions.
- A linear factor (ax+b) gives a partial fraction of the form $\frac{A}{ax+b}$.
- Factors $(ax+b)^2$ give partial fractions $\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$.
- Factors $(ax+b)^3$ give partial fractions $\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$.
- A quadratic factor (ax^2+bx+c) gives a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$.

EXAMPLE 1. $\int \frac{x+1}{x^2-3x+2}dx$

$$\frac{x+1}{x^2-3x+2}=\frac{x+1}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$$

Multiply both sides by the denominator (x-1)(x-2).

x+1 = A(x-2) + B(x-1)

Let (x-1) = 0 → x = 1

∴ 2 = A(-1) + B(0) ∴ A = -2

Let (x-2) = 0 → x = 2

∴ 3 = A(0) + B(1) ∴ B = 3

So the integration can now be written as:

$$\int \frac{x+1}{x^2-3x+2}dx=\int \frac{3}{x-2}dx-\int \frac{2}{x-1}dx=3\ln(x-2)-2\ln(x-1)+c$$

EXAMPLE 2. $\int \frac{x^2}{(x+1)(x-1)^2}dx$

$$\frac{x^2}{(x+1)(x-1)^2}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$$

Multiply both sides by the denominator $(x+1)(x-1)^2$.

$X^2 = A(x-1)^2 + B(x+1)(x-1)+C(x+1)$

Let (x-1) = 0 → x = 1

∴ 1 = A(0) + B(0)+C(2) ∴ C = 1/2

Let (x+1) = 0 → x = -1

∴ 1 = A(4) + B(0)+C(0) ∴ A = 1/4

Choose the highest power involved, i.e. x^2 in this example.

∴ 1 = A +B → B = 1 – A = 1 – 1/4 = 3/4

$$\therefore \int \frac{x^2}{(x+1)(x-1)^2} = \frac{1}{4}\int \frac{1}{x+1}dx + \frac{3}{4}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{(x-1)^2}dx$$

$$= \frac{1}{4}\ln(x+1) + \frac{3}{4}\ln(x-1) - \frac{1}{2(x-1)} + c$$

NOW DETERMINE THE FOLLOWING INTEGRATION

1) $\int \frac{x^2+1}{(x+2)^3}dx$

2) $\int \frac{4x^2+1}{x(2x-1)^2}dx$

3) $\int \frac{2x-1}{x^2-8x+15}dx$

4) $\int \frac{x^2}{x+1}dx$

5) $\int_0^{\pi} (\pi - x)\cos x dx$

6) $\int_0^{\pi} e^{2x}\cos 4x dx$

7) $\int_0^{\pi/6} e^{2\theta}\cos 3\theta d\theta$

8) $\int \tan^2 x \sec^2 x dx$

9) $\int \frac{dx}{\sqrt{x^2+4x+4}}$

10) $\int \frac{x-1}{9x^2-18x+17}dx$

11) $\int \frac{4x^5}{x^4-1}dx$

12) $\int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}d\theta$

13) $\int_1^2 (x-1)^2 \ln x dx$

14) $\int_0^{\pi/3} \frac{\sin x}{(1+\cos x)^2}dx$

FIRST ORDER DIFFERENTIAL EQUATIONS:-

A differential equation is a relationship between an independent variable ,x, a dependent variable ,y , and one or more differential coefficients of y with respect to x.

Examples:- $(x+3y)^3 \dfrac{d^2y}{dx^2} + 2 = 0$.

$$xy\frac{d^2y}{dx^2} + y\frac{dy}{dx} + e^{3x} = 0$$

Differential equations represent dynamic relationships , i.e. quantities that change, and are thus frequently occurring in scientific and engineering problems.

The **order** of a differential equation is given by the highest derivative involved in the equation.

$x\dfrac{dy}{dx} - y^2 = 0$ is a differential equation of the first order.

$xy\dfrac{d^2y}{dx^2} + y\dfrac{dy}{dx} + e^{3x} = 0$ is a differential equation of the second order.

$\dfrac{d^3y}{dx^3} - y\dfrac{dy}{dx} + \sin x \cos 4x = 8$ is a differential equation of the third order.

Similarly

1) $x\dfrac{dy}{dx} = y^2 + 6$ is a ………….. order equation.

2) $\cos^2x\dfrac{d^3y}{dx^3} - 5x\dfrac{dy}{dx} + 8 = 0$ is a ………….. order equation.

3) $y^4 + 1 = x - \dfrac{d^2y}{dx^2}$ is a ………….. order equation.

FORMATION OF DIFFERENTIAL EQUATIONS:-

Mathematically, they can occur when arbitrary constants are eliminated from a given function.

EXAMPLE 1. Consider y = Asinx + Bcosx , Where A , B are two arbitrary constants.

If we differentiate , we get $\dfrac{dy}{dx}$ = Acosx – Bsinx

And $\dfrac{d^2y}{dx^2}$ = - Asinx + Bcosx

Which is identical to the original equation , but with the sign changed .

i.e. $\dfrac{d^2y}{dx^2} = -y$ $\quad\therefore\quad \dfrac{d^2y}{dx^2} + y = 0$.

This is a differential equation of the second order.

EXAMPLE 2. Form a differential equation from the function $y = x + \dfrac{A}{x}$.

We have $y = x + \dfrac{A}{x} = x + Ax^{-1}$

$$\therefore \quad \frac{dy}{dx} = 1 - Ax^{-2} = 1 - \frac{A}{x^2}$$

From the given equation , $\frac{A}{x}$ = y – x , ∴ A = x(y – x) , ∴ $\frac{dy}{dx}=\frac{2x-y}{x}$

∴ x $\frac{dy}{dx}$ = 2x – y , and this is a differential equation of the first order.

IF we were to investigate the following, we should also find that :-

Y = Axe^x gives the diff. equation x $\frac{dy}{dx}$ = 2x – y (1st order)

Y = Ae^{-4x} + Be^{-4x} gives the diff. equation $\frac{d^2y}{dx^2}$ +10 $\frac{dy}{dx}$ + 24y = 0 (2nd order).

A function with 1 arbitrary constant gives a 1st order equation .

A function with 2 arbitrary constants gives a 2nd order equation .

So, without working each out in detail, we can say that

- Y = e^{-2x}(A + Bx) would give a diff. equation of ………… Order.
- Y = A $\frac{x-1}{x+1}$ would give a diff. equation of ………… Order.
- Y = e^{3x}(Acos3x + Bsin3x) would give a diff. equation of ………… Order.

Similarly

- X^2 $\frac{dy}{dx}$ + y = 1 is derived from a function having ………….. arbitrary constants.
- Cos^{2}x $\frac{dy}{dx}$ = 1 – y is derived from a function having ………….. arbitrary constants.
- $\frac{d^2y}{dx^2}$ + 4y $\frac{dy}{dx}$ + y = e^{2x} is derived from a function having ………….. arbitrary constants.

An nth order differential equation is derived from a function having n arbitrary constants.

SOLUTION OF DIFFERENTIAL EQUATIONS
METHOD 1. BY SEPARATING THE VARIABLES

A first order differential equation can be solved by this method if it is possible to collect all y terms with dy , and all x terms with dx as :
f(Y)dy = f(x)dx , then this equation can be solved by simple integration.

EXAMPLE 1. Solve $\frac{dy}{dx}=\frac{2x}{y+1}$

We can re-write this as $(y+1)dy=2xdx$

Now integrate both sides $\int(y+1)dy=\int 2xdx$

$$\therefore \quad \frac{(y+1)^2}{2}=x^2+c$$

EXAMPLE 2. Solve the differential equation $\frac{dy}{dx}=(1+x)(1+y)$

$$\frac{dy}{1+y}=(1+x)dx$$

$$\int\frac{dy}{1+y}=\int(1+x)dx$$

$$\therefore \quad \text{Ln}(1+y) = \frac{(1+x)^2}{2}+c$$

EXAMPLE 3. Solve the differential equation $\frac{dy}{dx}=\frac{y^2+xy^2}{x^2y-x^2}$

$$\frac{dy}{dx}=\frac{y^2(1+x)}{x^2(y-1)}$$

$$\frac{(y-1)dy}{y^2}=\frac{(1+x)dx}{x^2}$$

$$\int(\frac{1}{y}-\frac{1}{y^2})dy=\int(\frac{1}{x^2}+\frac{1}{x})dx$$

$$\therefore \quad \ln y+\frac{1}{y}=\ln x-\frac{1}{x}+c$$

REVISION EXERCISE. Find the general solution of the following equations:-

1) $y\tan x\frac{dy}{dx}=(4+y^2)\sec^2 x$

2) $\frac{dy}{dx}=\frac{y}{x}$

3) $\frac{dy}{dx}=(y+2)(x+1)$

4) $\cos^2 x\frac{dy}{dx}=(y+3)$

5) $\frac{dy}{dx}=xy-y)$

6) $\frac{\sin x}{1+y}\frac{dy}{dx}=\cos x$

METHOD 2. BY HOMOGENEOUS EQUATIONS – BY SUBSTITUTING Y = VX

This is determined by the fact that the total degree in x and y for each of the terms involved is the same . The key to solving every homogeneous equation is to substitute y = vx where v is a function of x. This converts the equation into a form in which we can solve by separating the variables.

IF we put y = vx , then by differentiate it with respect to x, given $\frac{dy}{dx}=v.1+x\frac{dv}{dx}=v+x\frac{dv}{dx}$

then We substitute them in the given equation.

EXAMPLE 1. Solve $\frac{dy}{dx} = \frac{x+3y}{2x}$

To solve this equation :- suppose that y = vx then differentiate it with respect to x , we get

$$\frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

The equation now becomes $v + x\frac{dv}{dx} = \frac{x+3vx}{2x} = \frac{x(1+3v)}{2x} = \frac{1+3v}{2}$

$$x\frac{dv}{dx} = \frac{1+3v}{2} - v = \frac{1+3v-2v}{2} = \frac{1+v}{2}$$

$\therefore x\frac{dv}{dx} = \frac{1+v}{2}$, this equation is now expressed in terms of v and x , we can solve it by separating the variables .

$$\int \frac{2}{1+v} dv = \int \frac{1}{x} dx$$

$\therefore$ 2ln(1+v) = lnx + c = lnx + lnA

$\therefore$ $\ln(1+v)^2 = \ln Ax$

$\therefore$ $(1+v)^2 = Ax$, but y = vx $\therefore v = \frac{y}{x}$

Which gives $(x+y)^2 = Ax^3$

EXAMPLE 2. Solve the differential equation $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

Here, all terms on the R.H.S. are of degree 2, i.e. the equation is homogeneous

$\therefore$ We substitute y = vx , $\therefore$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$, and the equation now becomes

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2 x^2}{v x^2} = \frac{x^2(1+v^2)}{v x^2} = \frac{1+v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{v} - v = \frac{1+v^2-v^2}{v} = \frac{1}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1}{v}$$

$$\int v\,dv = \int \frac{1}{x} dx$$

$\frac{v^2}{2} = \ln x + c$, but y = vx $\therefore v = \frac{y}{x}$

$$\therefore \quad \frac{1}{2}.\frac{y^2}{x^2} = \ln x + c$$

$\therefore$ $y^2 = 2x^2(\ln x + c)$

EXAMPLE 3. Solve the differential equation $(x^2+y^2)\frac{dy}{dx}=xy$

$$\frac{dy}{dx}=\frac{xy}{(x^2+y^2)}$$

$\therefore$ We substitute y = vx , $\therefore\ \frac{dy}{dx}=v+x\frac{dv}{dx}$, and the equation now becomes

$$v+x\frac{dv}{dx}=\frac{vx^2}{x^2+v^2x^2}=\frac{vx^2}{x^2(1+v^2)}=\frac{v}{1+v^2}$$

$$x\frac{dv}{dx}=\frac{v}{1+v^2}-v=\frac{v-v-v^3}{1+v^2}=\frac{-v^3}{1+v^2}$$

$$\therefore x\frac{dv}{dx}=\frac{-v^3}{1+v^2}$$

$$\int\frac{1+v^2}{v^3}dv=-\int\frac{1}{x}dx$$

$$\int(v^{-3}+v^{-1})dv=-\int x^{-1}dx$$

$$\frac{v^{-2}}{-2}+\ln v=-\ln x+c$$

$$\frac{v^{-2}}{-2}+\ln v=-\ln x+\ln A$$

$$\ln v+\ln x+\ln k=\frac{1}{2v^2}$$

$\therefore\quad \ln vxk=\frac{1}{2v^2}$, but y = vx $\therefore v=\frac{y}{x}$

$$\therefore \ln\frac{y}{x}xk=\frac{x^2}{2y^2}$$

$$\therefore 2y^2\ln yk=x^2$$

REVISION EXERCISE. Find the general solution of the following equations:-

1. $(x-y)\frac{dy}{dx}=x+y$
2. $2x^2\frac{dy}{dx}=x^2+y^2$
3. $(x^2+xy)\frac{dy}{dx}=xy-y^2$
4. $(2y-x)\frac{dy}{dx}=2x+y, given\ y=3, when\ x=2$
5. $(x^3+y^3)=3xy^2\frac{dy}{dx}$
6. $y-3x+(4y+3x)\frac{dy}{dx}=0$

METHOD 4. LINEAR EQUATIONS --- USE OF INTEGRATION FACTOR

The equation of the form $\frac{dy}{dx} + py = Q$, where p and Q are functions of x(or constants). This equation is called A LINEAR EQUATION OF FIRST ORDER and to solve any such equation, we multiply both sides by AN INTEGRATIHG FACTOR which is always I.f. = $e^{\int pdx}$ .This converts the L.H.S. into a complete differential coefficient of y.(I.f.)

EXAMPLE 1. Consider the equation $\frac{dy}{dx} + 5y = e^{2x}$

If we compare this equation with $\frac{dy}{dx} + py = Q$, we see that in this case p = 5 , Q = e^{2x} ,

$\therefore I.f. = e^{\int 5dx} = e^{5x}$, we begin by multiplying both sides by e^{5x}

$$e^{5x}\frac{dy}{dx} + 5ye^{5x} = e^{2x}e^{5x}$$

We now find the L.H.S. is the differential coefficient of ye^{5x} $\therefore \frac{d}{dx}\{y.e^{5x}\} = e^{7x}$

By integrate both sides W.R.T.x, $\therefore ye^{5x} = \int e^{7x}dx = \frac{e^{7x}}{7} + c$

$$\therefore y = \frac{e^{2x}}{7} + ce^{-5x}$$

EXAMPLE 2. Solve the differential equation $\frac{dy}{dx} - y = x.$

If we compare this equation with $\frac{dy}{dx} + py = Q$, we see that in this case p = -1 , Q = x ,

$\therefore I.f. = e^{\int -1dx} = e^{-x}$, we begin by multiplying both sides by e^{-x}

$$e^{-x}\frac{dy}{dx} - ye^{-x} = xe^{-x}$$

We now find the L.H.S. is the differential coefficient of ye^{-x} $\therefore \frac{d}{dx}\{y.e^{-x}\} = xe^{-x}$

By integrate both sides W.R.S. X ,

$$\therefore ye^{-x} = \int xe^{-x}dx = x(-e^{-x}) + \int e^{-x}dx = -xe^{-x} - e^{-x} + c$$

$\therefore y = -x - 1 + ce^{x}$, $\therefore y = Ce^{x} - x - 1$

VERY IMPORTANT NOTES FOR YOU

$e^{\ln(function)} = function$ such that

- $e^{\ln x} = x$
- $e^{\ln \sin x} = \sin x$
- $e^{2\ln x} = e^{\ln x^2} = x^2$

- $e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$
- $e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

EXAMPLE 3. Solve the differential equation $x\frac{dy}{dx} + y = x^3$

$\frac{dy}{dx} + \frac{1}{x}y = x^2$, p = 1/x , Q = x^2 , $\therefore I.f. = e^{\int \frac{1}{x}dx} = x$

$$x\frac{dy}{dx} + x.\frac{1}{x}y = x^3$$

$$\frac{d}{dx}(x.y) = x^3$$

$$yx = \int x^3 dx = \frac{x^4}{4} + c$$

$$\therefore yx = \frac{x^4}{4} + c$$

EXAMPLE 4. Solve the differential equation $\frac{dy}{dx} + y\cot x = \cos x$

If we compare this equation with $\frac{dy}{dx} + py = Q$, we see that in this case p =cotx , Q =cosx ,

$\therefore I.f. = e^{\int \cot dx} = e^{\int \frac{\cos x}{\sin x}dx} = e^{\ln \sin x} = \sin x$, $\therefore I.f. = \sin x$

Multiply both sides of the equation by sinx , we get :

$$\sin x\frac{dy}{dx} + y\cot x \sin x = \cos x \sin x$$

$(y.\sin x) = \int \sin x \cos x dx = \frac{\sin^2 x}{2} + c$, $\therefore y = \frac{\sin x}{2} + c\csc x$

REVISION EXERCISE. Find the general solution of the following equations:-

1. $(x+1)\frac{dy}{dx} + y = (x+1)^2$
2. $(x-2)\frac{dy}{dx} - y = (x-2)^3$
3. $x\frac{dy}{dx} + y = x\sin x$
4. $x\frac{dy}{dx} - 5y = x^7$
5. $\frac{dy}{dx} + 3y = e^{4x}$
6. $\tan\frac{dy}{dx} + y = \sec x$

METHOD 5. EXACT EQUATIONS

In this method , we use the idea of the total differential of function contains two variables. If the function W has continuous first partial derivative , then

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy$$

$\because \frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}$ functions depend on the same variables to the function W , we can write the above differential equation as : $dW = Q(x,y)dx + P(x,y)dy$

Such that $\frac{\partial W}{\partial x} = Q(x,y)$, $\frac{\partial W}{\partial y} = P(x,y)$

If dw = 0 , then W(x,y = c , by differentiate this equation with respect to x , we get

$\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y}.\frac{dy}{dx} = 0$, this is differential equation and the solution of it given by W(x,y = c

On the opposite side , if we have the differential equation $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$,

and we suppose the function W(x,y)=c is a general solution to the equation $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$ ……….. (1) such that $\frac{\partial W}{\partial x} = P(x,y)$, $\frac{\partial W}{\partial x} = P(x,y)$,

and if $y = \phi(x)$ is a differentiable function with respect to x , then

$$P(x,y) + Q(x,y)\frac{dy}{dx} = \frac{\partial W}{\partial x} + \frac{\partial W}{\partial y}.\frac{dy}{dx} = \frac{d}{dx}\{W(x,\phi(x)\} \qquad \text{………… (2)}$$

From equations (1) and (2) , we get $\frac{d}{dx}\{W(x,\phi(x)\} = 0$

Always the equation (1) is written as P(x,y)dx + Q(x,y)dy = 0 .

DEFINITION The deferential equation is said to be exact if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial X}$.

EXAMPLE 1. Find the general solution to the equation $2xy^3 + 3x^2y^2\frac{dy}{dx} = 0$

We observe the left side is the derivative of the function x^2y^3 with respect to x ,

$\therefore$ we can write the differential equation as $\frac{d}{dx}(x^2y^3) = 0$, and the general solution is given by the equation x^2y^3 = c .

EXAMPLE 2. Find the general solution to the equation $3x(xy-2) + (x^3 + 2y)\frac{dy}{dx} = 0$

To solve this equation we write it as $3x(xy-2)dx + (x^3 + 2y)dy = 0$, and compare it with P(x,y)dx + Q(x,y)dy = 0 .

$\therefore \frac{\partial W}{\partial x} = P(x,y)$ = 3x(xy-2) …………….. (1)

$\frac{\partial W}{\partial y} = Q(x,y) = x^3 + 2y$ …………….. (2)

We integrate (1) with respect to x , y is a constant $W = x^3y - 3x^2 + c(y)$ ………… (3)

To find c(y) , we differentiate it with respect to y and equal it with $\frac{\partial w}{\partial y}$ in (2) .

$\therefore \frac{\partial w}{\partial y} = x^3 + c^/(y) \Rightarrow x^3 + 2y = x^3 + c^/(y)$

$\therefore c^/(y) = 2y \Rightarrow c = y^2$ and substitute in (3) , we get the general solution

$W = x^3 y - 3x^2 + y^2$ and from the definition of the general solution we get

$x^3 y - 3x^2 + y^2 = c$.

Or We integrate (2) with respect to y , x is a constant $W = x^3 y + y^2 + c(x)$ ………… (4)

To find c(x) , we differentiate it with respect to x and equal it with $\frac{\partial w}{\partial x}$ in (1) .

$\therefore \frac{\partial w}{\partial x} = 3x^2 + c^/(x) \Rightarrow 3x^2 y - 6x = 3x^2 y + c^/(x)$

$\therefore c^/(x) = -6x \Rightarrow c = -3x^2$ and substitute in (4) , we get the general solution

$W = x^3 y + y^2 - 3x^2$ and from the definition of the general solution we get

$x^3 y - 3x^2 + y^2 = c$

EXAMPLE3. Solve the equation $(2x^3 - xy^2 - 2y + 3)dx - (x^2 y + 2x)dy = 0$

compare it with P(x,y)dx + Q(x,y)dy = 0 .

$\therefore \frac{\partial W}{\partial x} = P(x,y) = 2x^3 - xy^2 - 2y + 3$ …………….. (1)

$\frac{\partial W}{\partial y} = Q(x,y) = -(x^2 y + 2x)$ …………….. (2)

We integrate (1) with respect to x , y is a constant $W = x^3 y - 3x^2 + c(y)$ ………… (3)

To find c(y) , we differentiate it with respect to y and equal it with $\frac{\partial w}{\partial y}$ in (2) .

$\therefore \frac{\partial w}{\partial y} = x^3 + c^/(y) \Rightarrow x^3 + 2y = x^3 + c^/(y)$

$\therefore c^/(y) = 2y \Rightarrow c = y^2$ and substitute in (3) , we get the general solution

$W = x^3 y - 3x^2 + y^2$ and from the definition of the general solution we get

$x^3 y - 3x^2 + y^2 = c$.

Or We integrate (2) with respect to y , x is a constant $W = x^3 y + y^2 + c(x)$ ………… (4)

To find c(x) , we differentiate it with respect to x and equal it with $\frac{\partial w}{\partial x}$ in (1) .

$\therefore \frac{\partial w}{\partial x} = 3x^2 + c^/(x) \Rightarrow 3x^2 y - 6x = 3x^2 y + c^/(x)$

$\therefore c'(x) = -6x \Rightarrow c = -3x^2$ and substitute in (4) , we get the general solution

$W = x^3 y + y^2 - 3x^2$ and from the definition of the general solution we get

$x^3 y - 3x^2 + y^2 = c$

REVISION EXERCISE. Find the general solution of the following equations:-

1. $(x + y^2)\dfrac{dy}{dx} + (y + x) = 0$
2. $(2xy + 1)dx + (x^2 + 4y)dy = 0$
3. $(x\csc y - 2y)dy + (x + \sin y)dx = 0$
4. $(\tan x + 2y)\dfrac{dy}{dx} + (y\sec^2 x + \sec x \tan x) = 0$
5. $\left(\dfrac{2s-1}{t}\right)ds + \left(\dfrac{s - s^2}{t^2}\right)dt = 0$
6. $\ln y dx + x y^{-1} dy = 0$
7. $x(x^2 + y^2)dx + y(x^2 + y^2)dy = 0$
8. $\left(3e^{3x} - 2x\right)dx + e^{3x} dy = 0$
9. $\dfrac{(1 + y^2)ydx + (1 + x^2)xdy}{(1 + x^2 + y^2)^{3/2}} = 0$
10. $(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$
11. Is the differential equation $(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0$ an exact equation ?
12. Is the differential equation $\ln(y^2 + 1)dx + \dfrac{2y(x-1)}{x^2 + 1}dy = 0$ an exact equation ?
13. Form the differential equation if $y = e^{-kt}(A\cos nt + B\sin nt)$.
14. Form the differential equation if $y = (A + Bx)e^{nx}$.
15. Form the differential equation if $y = -9y^3 + C$.
16. Form the differential equation if $y^2 = Ax + B$.
17. Form the differential equation if $y^3 = Ax + A^2$.

NUMERICAL INTEGRATION (SIMPSON' S RULE)

$$\int_a^b f(x)dx = \frac{h}{3}\begin{bmatrix} f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)+....f(x_{n-1})\}+ \\ 2\{f(x_2)+f(x_4)+f(x_6)+....+f(x_{n-2})\}+f(x_n) \end{bmatrix}$$

Where $h=\frac{b-a}{n}$, n = عدد تقسيمات الفترة , $x_0=a$, $x_1=x_0+h$, $x_2=x_1+h$, …………, $x_n=x_{n-1}+h=b$.

EXAMPLE 1. Find the numerical integration $\int_1^{1.3}\sqrt{x}dx$ by Simpson's rule , n = 6 .

The solution

$h=\frac{b-a}{n}=\frac{1.3-1}{6}=0.05$, , $x_0=a=1$, $x_1=x_0+h=1+0.05=1.05$, $x_2=x_1+h=1.05+0.05=1.10$, $x_3=x_2+h=1.10+0.05=1.15$, $x_4=x_3+h=1.15+0.05=1.20$, $x_5=x_4+h=1.20+0.05=1.25$, $x_6=x_5+h=1.25+0.05=1.30=b=x_n$,

$$\int_a^b f(x)dx = \frac{h}{3}\begin{bmatrix} f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)+....f(x_{n-1})\}+ \\ 2\{f(x_2)+f(x_4)+f(x_6)+....+f(x_{n-2})\}+f(x_n) \end{bmatrix}$$

$$\therefore \int_1^{1.3}\sqrt{x}dx = \frac{h}{3}[f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)\}+2\{f(x_2)+f(x_4)\}+f(x_6)]$$

$$=\frac{0.05}{3}\left[\sqrt{1}+4\{\sqrt{1.05}+\sqrt{1.15}+\sqrt{1.25}\}+2\{\sqrt{1.10}+\sqrt{1.20}\}+\sqrt{1.30}\right]$$

= 0.01667[1+4{3.21511}+2{2.14426}+1.14018] = 0.32155

EXAMPL2. Find the numerical integration $\int_0^{\pi}\sin x dx$ by Simpson's rule , n = 6 .

The solution

$h=\frac{b-a}{n}=\frac{\pi-0}{6}=\frac{\pi}{6}$, $x_0=a=0$, $x_1=x_0+h=0+\frac{\pi}{6}=\frac{\pi}{6}$, $x_2=x_1+h=\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}$,

$x_3=x_2+h=\frac{\pi}{3}+\frac{\pi}{6}=\frac{\pi}{2}$, $x_4=x_3+h=\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}$, $x_5=x_4+h=\frac{2\pi}{3}+\frac{\pi}{6}=\frac{5\pi}{6}$,

$x_6=x_5+h=\frac{5\pi}{6}+\frac{\pi}{6}=\pi=b=x_n$,

$$\int_a^b f(x)dx = \frac{h}{3}\begin{bmatrix} f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)+....f(x_{n-1})\}+ \\ 2\{f(x_2)+f(x_4)+f(x_6)+....+f(x_{n-2})\}+f(x_n) \end{bmatrix}$$

$$\therefore \int_0^{\pi}\sin x dx = \frac{h}{3}[f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)\}+2\{f(x_2)+f(x_4)\}+f(x_6)]$$

$$=\frac{\pi}{18}\left[\sin 0+4\{\sin\frac{\pi}{6}+\sin\frac{\pi}{2}+\sin\frac{5\pi}{6}\}+2\{\sin\frac{\pi}{3}+\sin\frac{2\pi}{3}\}+\sin\pi\right]$$

$=\frac{\pi}{18}[0+4\{\frac{1}{2}+1+\frac{\sqrt{3}}{2}\}+2\{\frac{\sqrt{3}}{2}+\frac{1}{2}\}+0]$ = 1.3928

EXAMPL3. Find the numerical integration $\int_0^1 \sqrt{x^2+1}dx$ by Simpson's rule , n = 10 .

The solution

$h=\frac{b-a}{n}=\frac{1-0}{10}=0.1$, $x_0=a=0$, $x_1=x_0+h=0+0.1=0.1$, $x_2=x_1+h=0.1+0.1=0.2$,
$x_3=x_2+h=0.2+0.1=0.3$, $x_4=x_3+h=0.3+0.1=0.4$, $x_5=x_4+h=0.4+0.1=0.5$,
$x_6=x_5+h=0.5+0.1=0.6$, $x_7=x_6+h=0.6+0.1=0.7$, $x_8=x_7+h=0.7+0.1=0.8$
$x_9=x_8+h=0.8+0.1=0.9$, $x_{10}=x_9+h=0.9+0.1=1.0$

$$\int_a^b f(x)dx=\frac{h}{3}\begin{bmatrix} f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)+....f(x_{n-1})\}+ \\ 2\{f(x_2)+f(x_4)+f(x_6)+....+f(x_{n-2})\}+f(x_n) \end{bmatrix}$$

$$\therefore \int_0^1 \sqrt{x^2+1}dx=\frac{h}{3}\begin{bmatrix} f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)+f(x_7)+f(x_9)\}+ \\ 2\{f(x_2)+f(x_4)\}+f(x_6)+f(x_8)\}+f(x_{10})\} \end{bmatrix}$$

$$=\frac{0.1}{3}\begin{bmatrix} \sqrt{0+1}+4\{\sqrt{1.01}+\sqrt{1.09}+\sqrt{1.25}+\sqrt{1.49}+\sqrt{0.81}\}+ \\ 2\{\sqrt{1.04}+\sqrt{1.16}\}+\sqrt{1036}+\sqrt{1.81}\}+\sqrt{2} \end{bmatrix}$$

$$=\frac{0.1}{3}[1+22.932279+9.0887292+1.4142]$$

$$=\frac{0.1}{3}[34.435221]$$

= 1.1478407

EXAMPL4. Evaluate the numerical integration by Simpson's rule for the function which its values given below :

x	4	6	8	10
F(x)	1	3	8	20

The solution

$$h=\frac{b-a}{n}=\frac{10-4}{3}=2$$

$$\therefore \int_4^{10} f(x)dx=\frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+f(x_3)]$$

$$=\frac{2}{3}[1+12+16+20]$$

= 32.66666

EXAMPL5. Evaluate the numerical integration by Simpson's rule for the function which its values given below :

x	0	1	2	3	4	5
F(x)	0	16	48	67	88	0

The solution

$$h=\frac{b-a}{n}=\frac{2-0}{8}=0.25$$

$$\therefore \int_0^2 f(x)dx=\frac{h}{3}[f(x_0)+4\{f(x_1)+f(x_3)+f(x_5)+f(x_7)\}+2\{f(x_2)+f(x_4)+f(x_6\}+f(x_8)]$$

$$=\frac{0.25}{3}[1+4\{1.284+2.117+3.490+5.755\}+2\{1.649+2.718+4.482\}+7.380]$$

= 76.6629

REVISION EXERCISE.

Evaluate the following numerical integration :-

1) $\int_0^{\pi} e^x \cos x dx$

2) $\int_1^2 \ln x dx$

3)

x	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2
F(x)	1.000	1.284	1.649	2.117	2.718	3.490	4.482	5.755	7.380

DIFFERENCES

DEFINITION: The forward differences which denoted by $\Delta f(x_k) OR \Delta y_k$ is used to calculate near a tabular point x_k at the beginning of the tabulated range is given by :-

$\Delta f(x_k) = f(x_{k+1}) - f(x_k)$ OR written as $\Delta y_k = y_{k+1} - y_k$, which is called the first forward differences , such that the distance between any two sequence points is equal and like $x_{k+1} - x_k = h$.

$\Delta^2 f(x_k) = \Delta(\Delta f(x_k) = \Delta f(x_{k+1}) - \Delta f(x_k) = f(x_{k+2}) - 2f(x_{k+1}) + f(x_k)$, which is called the second forward differences , and the general form is $\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$, this is definition to the differences n .

The differences table is written as :-

x_k	y_k				
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

EXAMPLE1 Express $\Delta^2 y_0$, $\Delta^2 y_1$ with respect to the value of y .

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$

$= y_2 - 2y_1 + y_0$

$\Delta^2 y_1$=………….?

$\Delta^3 y_0$=………….?

EXAMPLE 2

Calculate to the fourth differences for the function :-

K	0	1	2	3	4	5	6	7
x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

k	x_k	y_k	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$	$\Delta^4 y_k$
0	1	1				
			7			
1	2	8		12		
			19		6	
2	3	27		18		0
			37		6	
3	4	64		24		0
			61		6	
4	5	125		30		0
			91		6	
5	6	216		36		0
			127		6	
6	7	343		42		
			169			
7	8	512				

FINDING POLYNOMIAL EQUATION BY NEWTON FORWARD

$$p(x_k) = y_k = \sum_{i=0}^{n} \binom{k}{i} \Delta^i y_0$$

$$= \frac{k!}{0!k!}\Delta^0 y_0 + \frac{k!}{1!(k-1)!}\Delta^1 y_0 + \dots\dots\dots\dots\dots + \frac{k!}{n!(k-n)!}\Delta^n y_0 + E_s \quad , \; k = \frac{x - x_0}{h}$$

EXAMPLE 1 Find a polynomial from degree 3 which taken the values from the table below:

k	0	1	2	3
x_k	4	6	8	10
y_k	1	3	8	20

The solution :-

k	x_k	y_k	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$
0	4	1			
			2		
1	6	3		3	
			5		4
2	8	8		7	
			12		
3	10	20			

$$p(x_k) = y_k = \sum_{i=0}^{3} \binom{k}{i} \Delta^i y_0$$

$$= y_0 + k\Delta^1 y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3 y_0$$

$$= 1 + 2k + 3\frac{k^2 - k}{2} + 4\frac{(k^2 - k)(k-2)}{6}\Delta^3 y_0$$

$$\because k = \frac{x - x_0}{h} = \frac{x-4}{2}$$

$$\therefore p(x_k) = 1 + (x-4) + \frac{3}{2}\left[\frac{(x-4)^2}{4} - \frac{(x-4)}{2}\right] + \frac{4}{6}\left[\frac{(x-4)^3}{8} - 3\frac{(x-4)^2}{4} + 2\frac{(x-4)}{2}\right]$$

$$= \frac{1}{24}(2x^3 - 27x^2 + 142x - 240)$$

EXAMPLE 2 Use Newton forward to find polynomial from degree 4 or less which take the values in the table below:-

k	0	1	2	3	4
x_k	1	2	3	4	5
y_k	1	-1	1	-1	1

k	x_k	y_k	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$	$\Delta^4 y_k$
0	1	1				
			-2			
1	2	-1		4		
			2		-8	
2	3	1		-4		16
			-2		8	
3	4	-1		4		
			2			
4	5	1				

$$p(x_k) = y_k = \sum_{i=0}^{4} \binom{k}{i} \Delta^i y_0$$

$$\therefore p(x_k) = \frac{1}{3}(2k^4 - 16k^3 + 40k^2 - 32k + 3)$$

$$\because k = \frac{x - x_0}{h} = \frac{x-1}{1} = x - 1$$

$$\therefore p(x_k) = \frac{1}{3}(2x^4 - 24x^3 + 100x^2 - 168x + 93)$$

EXAMPLE 3 Use Newton forward to find polynomial from degree 3 to the function :-
$y(x) = |x|$ at x= -2 , -1 , 0 , 1 , 2 .

The solution :-

k	x_k	y_k	Δy_k	$\Delta^2 y_k$	$\Delta^3 y_k$
0	-2	2			
			-1		
1	-1	1		0	
			-1		2
2	0	0		2	
			1		-2
3	1	1		0	
			1		
4	2	2			

$$p(x_k) = y_k = \sum_{i=0}^{3} \binom{k}{i} \Delta^i y_0$$

$$\therefore p(x_k) = = y_0 + k\,\Delta^1 y_0 + \frac{k(k-1)}{2!}\Delta^2 y_0 + \frac{k(k-1)(k-2)}{3!}\Delta^3 y_0$$

$$\because k = \frac{x - x_0}{h} = \frac{x+2}{1} = x+2$$

$$\therefore p(x_k) = \frac{1}{3}(x^3 + 3x^2 - x)$$

REVISION EXERCISE.

1) Use Newton forward to find polynomial from degree 4 or less which take the values in the table below:-

k	0	1	2	3	4
x_k	1	5/4	3/2	7/4	2
y_k	1	4/5	2/3	4/7	1/2

2) Use Newton forward to find polynomial from degree 3 to the function :-
$y(x) = \sin(\pi x / 2)$ at x= 0 , 1 , 2 ,3 .
Compare the two functions for the values x = 4 , x = 5 .
Is there exist a polynomial from degree 4 at x = 0 , 1 , 2 ,3 ,4 ,5 .

3) Find a polynomial which take the values below:-

$k=x_k$	0	1	2	3	4	5	6	7
y_k	1	2	4	7	11	16	22	29

ROOTS OF EQUATION

In order to find the roots of equation f(x) = 0 , we can use one of the methods :-

1) Iteration method .
2) Secant method .
3) Newton method .

1) ITERATION METHOD

Let f(x) = 0 , then take x = g(x) , find $g^{/}(x)$,

If $\left|g^{/}(x_0)\right| < 1$, then we can use x = g(x) , if not we change g(x) to another one .

That means we can find the approximate root by iteration $x_1 = g(x_0)$

$x_2 = g(x_1)$

$x_3 = g(x_2)$

$x_4 = g(x_3)$

.
.
.

$x_n = g(x_{n-1})$

such that $en = \left|x_n - x_{n-1}\right| \cong 0$, en is the error .

EXAMPLE 1 Find the root of the equation $f(x) = x^2 + 3x - 1$ by iteration method .

The solution :-

x = 1/(x+3)
$g(x) = (x+3)^{-1}$, $g^{/}(x) = -(x+3)^{-2}$
let $x_0 = 0.5$

$$\left|g^{/}(x_0)\right| = \left|-(0.5+3)^{-2}\right| = 0.08163 < 1$$

n	x_n
0	0.5
1	0.2857
2	0.3043
3	0.3026
4	0.3027
5	0.3027

∴ the root is 0.3027

EXAMPLE2 find the root of the equation f(x) = $x^2 - x - 3 = 0$.

The solution :-

$x = x^2 - 3$, then $g(x) = x^2 - 3$ and $g'(x) = 2x$.

let $x_0 = 2.3$, then $\left|g'(x_0)\right| = \left|2(2.3)\right| > 1$ this is wrong

let $x = (3/x)+1$, $g'(x) = -3/x^2$, $\left|g'(x_0)\right| = \left|\frac{-3}{(2.3)^2}\right| < 1$

n	x_n
0	2.3
1	2.3043
2	2.3019
3	2.3032
4	2.3025
5	2.3029
6	2.3027
7	2.3028
8	2.3028

$\therefore$ the root is 2.3028

2) **SECANT METHOD**

Let x_1 , x_2 two a proximate roots , we calculate x_3 from the law :-

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$, and

$$x_4 = x_3 - \frac{f(x_3)(x_3 - x_2)}{f(x_3) - f(x_2)} \quad , \quad x_5 = x_4 - \frac{f(x_4)(x_4 - x_3)}{f(x_4) - f(x_3)}$$, …… and so on to

$$x_{n+2} = x_{n+1} - \frac{f(x_{n+1})(x_{n+1} - x_n)}{f(x_{n+1}) - f(x_n)}$$ such that $\left|x_{n+2} - x_{n+1}\right| < \varepsilon$

$\therefore$ x_{n+2} is the a proximate root .

EXAMPLE1 Find the root of the equation f(x) = $x^6 - x - 1 = 0$ by secant method .

The solution :-

Let $x_1 = 1$, $x_2 = 2$

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} = 2 - \frac{(2^6 - 2 - 1)(2 - 1)}{(2^6 - 2 - 1) - (1^6 - 1 - 1)} = 1.016129$$

$x_4 = 1.190577$, $x_5 = 1.117655$, $x_6 = 1.132531$, $x_7 = 1.134816$,

$x_8 = 1.134723$, $x_9 = 1.134724$ $\therefore$ $x = 1.134724$

EXAMPLE 2 Find the root of the equation f(x) = x^2 - x -3 = 0 by secant method .

The solution :-

Let $x_1 = 2.2$, $x_2 = 2.3$

n	x_n
0	2.2
1	2.3
2	2.3285
3	2.30284
4	2.30279
5	2.30277
6	2.30277

$$\therefore \quad x = 2.30277$$

3) ***NEWTON METHOD***

$$x_1 = x_0 - \frac{f(x_0)}{f^{/}(x_0)} \quad , \ldots\ldots\ldots\ldots\ldots\ldots \quad , \quad x_n = x_{n-1} - \frac{f(x_{n-1})}{f^{/}(x_{n-1})}$$

EXAMPLE1 By using Newton-method ,find the root of the equation f(x) = x^6 - x -1 = 0 .

The solution :-

$\because$ f(1) = -1 and f(2) = +61 , Let $x_0 = 2$

$$f^{/}(x) = 6x^5 - 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f^{/}(x_0)} = 2 - \frac{(2^6 - 2 - 1)}{(6*2^5 - 1)} = 1.6806$$

n	x_n
0	2
1	1.6806
2	1.4307
3	1.2549
4	1.1615
5	1.1363
6	1.1347
7	1.1347

$$\therefore \quad x = 1.1347$$

EXAMPLE 2 By using Newton-method ,find the root of the equation f(x) = $x^3 + 4x^2 - 10 = 0$.

The solution :-

$\because$ f(1) = -5 and f(2) = +14 , Let $x_0 = 1.5$

$$f^{/}(x) = 3x^2 + 8x$$

$$x_1 = x_0 - \frac{f(x_0)}{f^{/}(x_0)} = 1.5 - \frac{(1.5^3 + 4*1.5^2 - 10)}{(3*1.5^2 + 8*1.5)} = 1.37333$$

n	x_n
0	1.5
1	1.37333
2	1.36526
3	1.36523
4	1.36523

$$\therefore \quad x = 1.36523$$